Lecture 15: Self-Adjointness

- Recall from math 54 that if $A = A^T$, A is called a symmetric matrix and we may apply the Spectral Theorem to diagonalize A:

e.g.) $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ has $A = A^T$

and A has eigenvalues that are roots of
$(2-x)^2 - 1 = x^2 - 4x + 3$ i.e. $x = 1$, $x = 3$.

The eigenvectors are $\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ giving

$$A = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$$

- Another way to view symmetry: on $\mathbb{R}^n$
$\langle Ax, y \rangle = (Ax) \cdot (y) = x \cdot (A^T y) = \langle x, A^T y \rangle$
So $A = A^T$ says $\langle Ax, y \rangle = \langle x, Ay \rangle$

- This is a property we may generalize: for a linear map $A : \mathcal{H} \to \mathcal{H}$, we will focus on self-adjoint cases where $\langle Ax, y \rangle = \langle x, Ay \rangle$ for all $x, y \in \mathcal{H}$.

$\boxed{\text{Lemma 7.11}}$ Suppose that $\Omega \subseteq \mathbb{R}^n$ is a bdd. domain with C^1 bdry. If $u, v \in C^2(\bar{\Omega})$ and both satisfy either homogeneous Dirichlet or Neumann bdry conditions, on $\partial\Omega$, then

$$\langle \Delta u, v \rangle = \langle u, \Delta v \rangle.$$

$\boxed{\text{Pf:}}$ $\int_\Omega (\Delta u)\bar{v} - u(\overline{\Delta v})\,dx = \int_{\partial\Omega} \left[\bar{v}\frac{\partial u}{\partial n} - u\frac{\partial \bar{v}}{\partial n} \right] dx$

and Dirichlet $v|_{\partial\Omega} = 0 = u|_{\partial\Omega}$
Neumann $\frac{\partial v}{\partial n}\big|_{\partial\Omega} = 0 = \partial u/\partial n\,\big|_{\partial\Omega}$

give RHS $= 0$ $\square$

• Thus, under appropriate space restrictions and Boundary Conditions, the Laplacian ~~[illegible]~~ acts self-adjoint (spaces/domains to make it self-adjoint are more nuanced).

$\boxed{\text{Lemma 7.12}}$ suppose $\{\lambda_j\}$ is a sequence of eigenvalues of $-\Delta$ on a bdd domain $U \subseteq \mathbb{R}^n$, with eigenvectors in $C^2(\bar{U})$ subject to a self-adjoint bdry condition.
Then, $\lambda_j \in \mathbb{R}$ and, after a possible rearrangement, the eigenvectors form an orthonormal sequence in $L^2(U)$.
Furthermore, $\lambda_j > 0$ for Dirichlet conditions and $\lambda_j \geq 0$ for Neumann.

$\boxed{\text{Pf}}$ Suppose we have a sequence $\{\phi_j\} \subseteq C^2(U)$ satisfying $-\Delta\phi_j = \lambda_j \phi_j$. Notice $\langle \Delta\phi_j, \phi_j \rangle = \langle \phi_j, \Delta\phi_j \rangle \Rightarrow (\lambda_j - \bar{\lambda}_j)\|\phi_j\|^2 = 0$ so $\lambda_j \in \mathbb{R}$.

Then, $\langle \phi_j, \Delta\phi_K \rangle = \langle \Delta\phi_j, \phi_K \rangle$

$\Rightarrow -\lambda_j \langle \phi_j, \phi_K \rangle = -\bar{\lambda}_K \langle \phi_j, \phi_K \rangle$

and for $\lambda_j \neq \lambda_K \Rightarrow \langle \phi_j, \phi_K \rangle = 0.$

If some λ_j's are equal, we use Gram-Schmidt to obtain orthonormal eigenvectors.

Normalizing so $\|\phi_j\|_2 = 1$

$$\lambda_j = \langle -\Delta\phi_j, \phi_j \rangle = \int_\Omega (-\Delta\phi_j)(\bar{\phi}_j)\,dx$$
$$= \int_\Omega |\nabla\phi_j|^2\,dx - \underbrace{\int_{\partial\Omega} \bar{\phi}_j \frac{\partial\phi_j}{\partial\eta}\,dS}_{\text{vanishes in Dirichlet/Neumann conditions.}}$$

If ~~[illegible]~~, $\lambda_j = 0$, $\nabla\phi_j = 0$ gives ϕ_j constant.
In the Dirichlet case, $\phi_j = 0$ is trivial and not an eigenvector. $\square$

ex.) In lecture 10, we found a set of eigenfunctions for $-\Delta$ on $\mathbb{D} \subseteq \mathbb{R}^2$, with Dirichlet B.C

$$\phi_{k,m}(r,\theta) = e^{ik\theta} J_k(j_{k,m} r) \qquad k \in \mathbb{Z},\ m \in \mathbb{N}$$

Corresponding to eigenvalue $j_{k,m}$
(no $j_{k,m}$ was repeated, as it was the 0 of J_k's, except possibly $j_{k,m} = j_{-k,m}$ b/c $J_{-k} = -J_k$)

•) The orthogonality condition is

$$\langle \phi_{k,m}, \phi_{k',m'} \rangle_{L^2} = \int_0^1 \int_0^{2\pi} \phi_{k,m}(r,\theta)\, \phi_{k',m'}(r,\theta)\, r\,d\theta\,dr$$

$$= \int_0^1 \int_0^{2\pi} e^{i(k-k')\theta} J_k(j_{k,m} r) J_{k'}(j_{k',m'} r)\, d\theta\, dr$$

if $k \neq k'$, $\int_0^{2\pi} e^{i(k-k')\theta} d\theta = 0$ so this whole integral is 0.

if $k = k'$

$$\langle \phi_{k,m}, \phi_{k',m'} \rangle = 2\pi \int_0^1 r\, J_k(j_{k,m} r) J_{k'}(j_{k',m'} r)\, dr$$

$$= 0 \quad \text{for } m \neq m'$$

by the orthog. of eigenfunctions.

This occurs because of oscillations in J_k